Jackson 2.26 (a)

According to Jackson (2.69) and (2.70), The most general solution for corner is

$$\Phi(\rho,\phi) = (a_0 + b_0 \ln\rho)(A_0 + B_0\phi) + \sum_{\nu>0} \left[a_\nu \rho^\nu + b_\nu \rho^{-\nu}\right]\left[A_\nu \cos(\nu\phi) + B_\nu \sin(\nu\phi)\right]$$

$\Phi|_{\phi=0,\beta} = 0$ demands ~~periodicity in~~ $\nu = \frac{n\pi}{\beta}$, and no cos terms appear.

$$\Rightarrow \Phi(\rho,\phi) = (a_0 + b_0\ln\rho)(A_0 + B_0\phi) + \sum_\nu \left[A_\nu \rho^\nu + B_\nu \rho^{-\nu}\right]\sin(\nu\phi),$$

$$\nu = \frac{n\pi}{\beta}.$$

$$\Phi|_{\phi=0,\beta} = 0 \Rightarrow a_0 = 0,\ b_0 = 0,\ A_0 = 0,\ B_0 = 0.$$

$$* \quad \Rightarrow \Phi(\rho,\phi) = \sum_n \left[A_n \rho^{n\pi/\beta} + B_n \rho^{-n\pi/\beta}\right] \sin\left(\frac{n\pi\phi}{\beta}\right)$$

This is the series representation of the solution, we now try to determine the coefficients.

We know $\Phi|_{\rho=a} = 0$, by orthogonality condition

$$\int_0^\beta \Phi|_{\rho=a} \sin\left(\frac{n\pi\phi}{\beta}\right) d\phi = 0 = \left[A_n a^{n\pi/\beta} + B_n a^{-n\pi/\beta}\right]\frac{\beta}{2}$$

$$\Rightarrow A_n a^{n\pi/\beta} + B_n a^{-n\pi/\beta} = 0,$$

$$B_n = -A_n a^{2n\pi/\beta}.$$

We are thus encouraged to modify our series solution:

$$\Phi(\rho,\phi) = \sum_n \left[A_n \rho^{n\pi/\beta} - A_n a^{2n\pi/\beta} \rho^{-n\pi/\beta}\right] \sin\left(\frac{n\pi\phi}{\beta}\right)$$

$\ast$ $$= \sum_n A_n \left[\rho^{n\pi/\beta} - a^{2n\pi/\beta} \rho^{-n\pi/\beta}\right] \sin\left(\frac{n\pi\phi}{\beta}\right).$$

To determine A_n we would have to know the boundary condition for large ρ, suppose we are given boundary condition $\Phi(c,\phi)$ for ~~⟺~~ $c>a$. Then by orthogonality.

$$\int_0^\beta d\phi\, \Phi(c,\phi) \sin\left(\frac{n\pi\phi}{\beta}\right) = \frac{\beta}{2} A_n \left[c^{n\pi/\beta} - a^{2n\pi/\beta} c^{-n\pi/\beta}\right]$$

$\ast$ $$\Rightarrow \quad A_n = \frac{2}{\beta}\left[c^{n\pi/\beta} - a^{2n\pi/\beta} c^{-n\pi/\beta}\right] \int_0^\beta d\phi\, \Phi(c,\phi) \sin\left(\frac{n\pi\phi}{\beta}\right).$$

Jackson 2.26 (b)

$$\Phi = \sum_n A_n \left[\rho^{\nu} - a^{2\nu} \rho^{-\nu}\right] \sin(\nu\phi) \qquad \nu = \frac{n\pi}{\beta}.$$

$$E_\phi = -\frac{\partial \Phi}{\partial \phi} = -\sum_n A_n \left[\rho^{\nu} - a^{2\nu}\rho^{-\nu}\right] \nu\cos(\nu\phi)$$

* Lowest nonvanishing term : $E_\phi \simeq -A_1\left[\rho^{\pi/\beta} - a^{2\pi/\beta}\rho^{-\pi/\beta}\right]\frac{\pi}{\beta}\cos\left[\frac{\pi\phi}{\beta}\right]$

$$E_\rho = -\frac{\partial \Phi}{\partial \rho} = -\sum_n A_n \left[\nu\rho^{\nu-1} - a^{2\nu}\rho^{-(\nu+1)}(-\nu)\right]\sin(\nu\phi)$$

$$= -\sum_n A_n \,\nu \left[\rho^{\nu-1} + a^{2\nu}\rho^{-(\nu+1)}\right]\sin(\nu\phi)$$

* Lowest nonvanishing term: $E_\rho \simeq -A_1 \frac{\pi}{\beta}\left[\rho^{\frac{\pi}{\beta}-1} + a^{2\pi/\beta}\rho^{-\left(\frac{\pi}{\beta}+1\right)}\right] \sin(\frac{\pi\phi}{\beta})$

To find surface charge density, use $E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$.

* $\sigma(\rho, 0) = \varepsilon_0 E_\phi\Big|_{\phi=0} \simeq -A_1\varepsilon_0\frac{\pi}{\beta}\left[\rho^{\pi/\beta} - a^{2\pi/\beta}\rho^{-\pi/\beta}\right]$

* $\sigma(\rho, \beta) = \varepsilon_0 E_\phi\Big|_{\phi=\beta} \simeq +A_1\varepsilon_0\frac{\pi}{\beta}\left[\rho^{\pi/\beta} - a^{2\pi/\beta}\rho^{-\pi/\beta}\right]$

* $\sigma(a, \phi) = \varepsilon_0 E_\rho\Big|_{\rho=a} \simeq -A_1\varepsilon_0\frac{\pi}{\beta}\left[a^{\frac{\pi}{\beta}-1} + a^{\frac{2\pi}{\beta}} a^{-\frac{\pi}{\beta}-1}\right]\sin(\frac{\pi\phi}{\beta})$

$$= -2A_1\varepsilon_0\frac{\pi}{\beta}a^{\frac{\pi}{\beta}-1}\sin\left(\frac{\pi\phi}{\beta}\right)$$

Jackson 2.26 (c) (first part)

For $\beta = \pi$, we have approximately,

$E_\phi \simeq -A_1 [\rho - a^2 \rho^{-1}]^{*} \cos\phi$

$E_\rho \simeq -A_1 [1 + a^2 \rho^{-2}] \sin\phi$

Dividing E_ϕ by ρ gives $E_\phi \simeq -A_1 [\cancel{\rho} 1 - a^2 \rho^{-2}] \cos\phi$.

For large ρ, we have $E_\phi \sim -A_1 \cos\phi \, \hat{\phi}$

$E_\rho \sim -A_1 \sin\phi \, \hat{\rho}$.

using $\hat{\phi} = -\sin\phi \, \hat{x} + \cos\phi \, \hat{y}$ $\quad \hat{\rho} = \cos\phi \, \hat{x} + \sin\phi \, \hat{y}$

$\Rightarrow \vec{E} \sim -A_1 \{ -\sin\phi\cos\phi \, \hat{x} + \cos^2\phi \, \hat{y} + \sin^2\phi \, \hat{y} + \sin\phi\cos\phi \, \hat{x} \}$

$= \boxed{-A_1 \hat{y}}$

This field is uniform, normal to the plane.

Davidson Cheng

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